Valuing streams of risky cash flows with risk-value models

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Abstract

Based on risk-value models we introduce a multi-period approach to the valuation of streams of risky cash flows. The valuation is based on the (expected) value of the output's or input's magnitude and the risk of the output cash flow, as captured by a risk measure. We derive three formulae for valuing single cash flows and utilize the principles of separate valuation and of cumulating the cash flows to derive a multi-period valuation method. In an axiomatic way, the article sets the foundations for a new approach and suggests several directions for its further development.

Keywords: Risk-value models, company valuation, project valuation, alternative approach

1. Introduction

The problem of valuing a stream of risky cash flows, in the sense of company or project valuation, is one of the most important economic tasks and of special importance during the preparatory stages of the decision-making process. In this paper, we suggest a new solution to the problem, which is based on risk measures such as the value-at-risk and an internal risk model of the firm. With our approach the value contribution of the risk management can be made explicit.

The standard valuation of risky cash flows based on the hypothesis of perfect and complete capital markets, especially when reverting to the Capital Asset Pricing Model (CAPM), is based on model

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assumptions that are easy to falsify, while empirical evidence in favor of these models is not very convincing (see e.g. Dempsey, 2013). Also the replication methodology, which is based on the no arbitrage argument and used in derivatives pricing, is not applicable in the context of a firm's investment decisions, as the related cash flows cannot be determined for all conceivable states of the world and capital markets are not complete, implying that a perfect replication of the cash flows is not possible in reality. Another strand of literature uses utility functions for the valuation of risky cash flows, which may also be of theoretical interest but, due to its empirical falsifiability (Tversky and Kahneman, 1992) and lack of acceptance by practitioners, can hardly be applied to real world valuation situations. In consideration of this, our paper presents a new methodology based on the 'incomplete replication' approach, which combines the advantages of the replication approach with realistic requirements, in which the informational situation is concerned. The incomplete replication methodology is based on the idea of the risk-value model of Sarin and Weber (1993), who, however, clearly stick to the tradition of expected utility and assume a valuing individual with a concrete utility function. Our approach is more general, as it builds on the reference investment possibilities available to the decision maker. By choosing a market index and a risk free investment (like government bonds), which is the idea pursued in this article, one can reconcile the purely individual concept with market-oriented valuation approaches. The approach presented here leaves some choices to be made by the decision maker (the risk measure and input vs. output perspective). Therefore, the risk preferences can be expressed by selecting these items according to the individual perception of the risk inherent in streams of cash flows. Our approach is self-consistent and built on the assumption that a decision maker uses μ-risk reasoning to compare risky cash flows. Our approach is not in any case compatible with utility-based valuation methods. Thus, a comparison with utility-based approaches may be interesting, but is, however, not a necessary preparatory step for establishing our approach.

A central advantage of the incomplete replication method lies in the fact that historical stock returns of the valuation object or comparable firms are not required for the appraisal, imperfections and 'anomalies' in the capital market are therefore irrelevant. The valuation is built on the stochastic characteristics of the cash flows to be appraised, which also enables its application to unlisted companies or real assets, especially for small and medium sized enterprises (SMEs). The informational requirements can, in particular, be covered by analyzing the frequency distribution generated through Monte Carlo (MC) simulation. Additionally, typical financing and rating restrictions can be taken into account through an appropriate choice of the risk measure. For example, if a restriction exists in the form of a maximum acceptable probability of default, appropriate equations can be deduced directly from the creditors' requirements concerning the value-at-risk (VaR), in which case reference is made to the equity needed to buffer possible losses. In contrast to previous research on valuation with risk-value models, we also tackle the realistic issue of simultaneously valuing several subsequent cash flows belonging to one investment.

Our valuation methodology uses the following assumptions: First, the decision maker faces an alternative investment, typically consisting of a riskless asset with interest rate r and a risky asset, e.g. the market portfolio. Second, the decision maker appraises on the basis of exactly two pieces of information, namely the expected value of the input's or the output's magnitude and a risk measure ρ of the output. In this context, the input and output quantities are cash flows, the first one known today and the second one stochastic. As a result we achieve a valuation of the single cash flows by risklessly discounted expected values diminished by a risk discount dependent on the risk measure. Our approach is not compatible with no-arbitrage pricing, which is unproblematic whenever arbitrage is practically unfeasible. This is exactly the setup in which the approach is supposed to be applied. In case of perfect replicability of the stream of cash flows the risk is completely eliminable and our approach loses its relevance, even if it can artificially be generalized to these cases.

Throughout this paper we will not distinguish between cash flows generated by a single project and those corresponding to a whole company. A difference between the valuation of both kinds could be relevant in a setting in which diversification with other assets is considered, a factor we neglect to present our new approach as clearly as possible. However, we discuss the treatment of this issue in the penultimate section. For maximal clarity we also do not optimize the project's cash flows by considering hedging possibilities and other means of altering the nature of the cash flows. We assume that the stream of cash flows is given and that the hedging decisions already have been taken. To this end, we follow the principle that valuing means comparing and not optimizing. Moreover, we have chosen to model the stream of cash flows without using a filtration. For the basic cases we treat in this article, which only employ non-dynamic strategies, this is sufficient. Therefore, for the ease of notation we refrain from using filtrations.

The remainder of the paper is organized as follows: First we give a short review of related literature. Then we develop our basic ideas in a one-period framework as a preparation step while the next section discusses various concepts of expanding the one-period framework to two or more periods, which is the actual case of interest. After an illustration of our approach by means of an example and a discussion on several important issues for practice and further research, the paper finishes with a conclusion.

2. Related literature

The value of the company or a project will only rarely directly correspond to its market price (see e.g. Shiller (1981); Lee et al. (1999); Brennan and Wang (2010)), if this entity exists at all, as is the case for listed companies. Therefore appraisal techniques are required to transform future risky cash flows into a present value.

This can be accomplished through various different techniques, the two main methods being discounted cash flow (DCF) or the very simplistic multiple methods (see a contemporary textbook like Berk and DeMarzo (2014) for a state-of-the-art presentation of both concepts). Such capital-market oriented valuation models require no utility function and are based on an idealized market calculation, for example, an arbitrage-free or perfect and complete capital market, enabling a perfect duplication of the cash flow stream to be valued. In particular, however, the necessity of fulfilling an infinite number of state conditions does not permit perfect replication of cash flows. The reason why the CAPM is so significant in capital-market oriented valuation theory is because it requires no completeness of the capital market while offering a simple formula for the discount interest rate, based on μ-σ preference functions. Yet, empirical research has shown that neither realized nor anticipated stock returns can be explained by means of the beta factor alone (see the critique by Dempsey (2013)). Alternatives such as the three-factor model of Fama and French (1993), the four-factor model of Carhart (1997) and the five-factor model of Fama and French (2015), are, in the valuation context, only appealing to a limited extent, as they lack a clear theoretical foundation (see, e.g. for empirical results, Fama and French (2008, 2012)). Recent empirical studies, for example by Zhang (2009), show that precisely those company-specific variables such as growth or return on equity explain stock returns better than variables derived from the capital market. This has led to several alternative three-factors models (see e.g. Walkshäusl and Lobe (2014)), which address the asset pricing theory of Cochrane (1991, 1996). Doubt as to a capital-market oriented valuation also arises from the distress and volatility anomalies indicated by Campbell et al. (2008), since less risky companies exhibit excess returns (Ang et al., 2006, 2009).

An alternative approach is to utilize individual preferences in the valuation, with one possibility being the application of utility functions. While in the English speaking literature this is a perspective which is adhered to rather rarely (see Smith (1998) and Hazen (2009) for comprehensive references), there was a vast debate in German business literature in the first decade of the 21st century on several variants of utility based cash flow valuation (see Schosser and Grottke (2013) for a summary of this debate). These valuation methods assume that the preferences of a decision maker can be described by a neoclassical utility function, a notion which has been questioned in the academic discussion for several decades (see e.g. Fishburn, 1977; Tversky and Kahneman, 1979). The valuation principle presented below will be derived completely from the decision space and the choice of a risk measure. Both the price of the risk as well as the riskless interest rate can be drawn from the investment alternative of the decision maker. Time preferences are not an explicit part of the model.

A formally different idea is the valuation based on certainty equivalents, as seen in Robichek and Myers (1966a,b); Kudla (1980), where—besides an arbitrary determination—the certainty equivalent can be derived either from the CAPM, from utility functions or from μ-risk preferences. While the first two versions are already covered by the literature cited above, the latter is the basis of this contribution. Direct forerunner works to our research are Spremann (2004), who derives the CAPM valuation of a single cash flow by means of (incomplete) replication and μ-σ preference functions, and Gleißner and Wolfrum (2009), who generalize the approach for position-invariant risk measures. Gleißner (2006) introduces a special variant of the one-period input-oriented framework, but without an axiomatic foundation. None of the three references treats general risk measures or a real multi-period valuation of cash flows streams.

3. One-period valuation

We start with the problem of appraising one single cash flow X_t, which realizes at time t. To simplify notation we fix t and omit the time index for the remainder of this section.[1] First, we consider the output-oriented view, which represents the classical risk-value model. As a second step, we move on to the input-oriented perspective in which not only the expected cash flow and its risk are considered but also the magnitude of the input needed.

3.1. Output-oriented view

The idea is to construct a reference investment, consisting of a riskless asset yielding an interest rate r (within the time interval $[0,t]$) and a risky asset, which we assume to be a market portfolio. The latter does not necessarily need to be the one of the capital asset pricing model (Sharpe, 1964; Lintner, 1965). It is just supposed to be a diversified risky stock investment, which serves as a general investment alternative for the investor who wishes to value the cash flow. Any combination of a riskless and a risky investment can be utilized as long as this represents a realistic alternative for the valuing subject. The only two properties that the reference investment has to meet are the following: both components need to be discretionarily scalable and they need to have a unique price today. Therefore it are mainly capital market instruments that recommend themselves for the

[1] In Section 4.1 we will treat the case that t is a variable quantity.

reference investment. If the valuation is carried out from an objective perspective the proxy for the risk-free asset may be chosen as a government bond of the best rating and an empirical market portfolio, presumably a broad stock index.

The return of the market portfolio within the interval $[0,t]$ is denoted by R_M, with expected value μ_M. The portfolio to be constructed is supposed to replicate the expected value and the risk (as measured by some suitable risk measure $\rho(.)$) of the cash flow X at time t. The reason why we call this approach *output-oriented* lies in the fact that the reference investment is calibrated in a way to (incompletely) replicate the output of the investment leading to cash flow X. This is done by investing a certain amount of capital y risklessly ($y < 0$ means taking a loan) and the amount of capital z in the market portfolio. If the (time-t) output of the reference investment is identical in terms of expected value and risk measure, the sum $y+z$ yields the desired valuation at time 0 of the cash flow X. By assumption and since this case is not needed for replication, we exclude the case of shorting the market portfolio, which implies $z \geq 0$.

The approach results in the two equations

$$\mathrm{E}(X) = \mathrm{E}(y(1+r)+z(1+R_M)) = y(1+r)+z(1+\mu_M) \tag{1}$$

and

$$\rho(X) = \rho(y(1+r)+z(1+R_M)). \tag{2}$$

Concerning the risk measure, many possibilities can be considered (Emmer et al., 2015). For example it could be the standard deviation as well as the value-at-risk (VaR) or the conditional value-at-risk (CVaR), to name the most popular ones, but also more a more advanced risk measure such as the modified expected shortfall (Jadhav et al., 2013). In the theory of risk measures some possible properties are well-known. In our context the three following properties are of interest. Let $\mathscr{X}$ be the set of all real-valued random variables on some probability space $(\Omega, \Xi, \mathscr{P})$. Then

- positive homogeneity (PH) is defined by $\rho(aX) = a\rho(X)$ for all $X \in \mathscr{X}$ and $a \geq 0$,

- translation invariance (TI) by $\rho(X+a) = \rho(X) - a$ for all $X \in \mathscr{X}$ and $a \in \mathbb{R}$,

- position invariance (PI) by $\rho(X+a) = \rho(X)$ for all $X \in \mathscr{X}$ and $a \in \mathbb{R}$.

Note that while coherent risk measures (see Artzner et al., 1999) fulfill PH and TI and also the axioms of subadditivity and monotonicity, we do not restrict our approach to these risk measures. The following theorem presents a valuation formula for risk measures, which fulfill PH and at the same time either TI or PI.

Theorem 1 *If the risk measure ρ with $\rho\left(X - E(X)\right) \geq 0$ and $\rho\left(R_M - \mu_M\right) > 0$ fulfills PH and TI or PI, then today's value of X according to the output-oriented view is given by:*

$$V(X) = \frac{E(X) - \rho\left(X - E(X)\right) \frac{\mu_M - r}{\rho(R_M - \mu_M)}}{1+r} \tag{3}$$

Proof. We commence with equations (1) and (2). If we solve (1) for y, we receive

$$y = \frac{\mathrm{E}(X) - z(1+\mu_M)}{1+r}.$$

This term plugged into (2) yields $\rho\left(X\right) = \rho\left(z(R_M - \mu_M) + \mathrm{E}(X)\right)$. Now we distinguish two cases.
1. case: If ρ fulfills PH and TI, we have $\rho\left(X\right) + \mathrm{E}(X) = z\rho\left(R_M - \mu_M\right)$ and thus receive:

$$z = \frac{\rho\left(X\right) + \mathrm{E}(X)}{\rho\left(R_M - \mu_M\right)} = \frac{\rho\left(X - \mathrm{E}(X)\right)}{\rho\left(R_M - \mu_M\right)}.$$

2. case: If ρ fulfills PH and PI, we have $\rho\left(X\right) = z\rho\left(R_M\right)$ and thus receive:

$$z = \frac{\rho\left(X\right)}{\rho\left(R_M\right)} = \frac{\rho\left(X - \mathrm{E}(X)\right)}{\rho\left(R_M - \mu_M\right)}.$$

In either case an amount of $z = \frac{\rho(X - \mathrm{E}(X))}{\rho(R_M - \mu_M)} \geq 0$ is invested in the market portfolio. The value of y can be derived by plugging z into the first of the above equations, yielding:

$$y = \frac{\mathrm{E}(X) - \frac{\rho(X - \mathrm{E}(X))}{\rho(R_M - \mu_M)}(1+\mu_M)}{1+r}.$$

The sign of this term is not positive in any case, but can also be minus. Adding up y and z establishes the claim. $\square$

In the following we discuss several aspects of Theorem 3.1.

If we assume that $\mu_M > r$, which is characteristic for a market with risk averse agents, the derived valuation formula is a *certainty equivalent method* (Kudla, 1980), meaning that the expected value of X is diminished by a term depending on the risk of X, namely $\rho(X - \mathrm{E}(X))$. Note that it is now the risk of the *centered variant* of X that is used, which comes as a result and not a prerequesite of our approach. This kind of certainty equivalent is then discounted with the riskfree interest rate. The risk term is multiplied with

$$\lambda^O := \frac{\mu_M - r}{\rho(R_M - \mu_M)}. \tag{4}$$

One can interpret this term as a risk premium which the investment has to yield for every unit of (centered) risk taken.

Note that even if we do not use a risk measure ρ with the PI property, e.g. the VaR, we end up with measuring the risk of the centered cash flow $X - \mathrm{E}(X)$ and thus with the PI version of the risk measure, e.g. the DVaR. If we use $\rho(.) = \mathrm{Cov}(., R_M)$ as a risk measure, this exactly depicts the results of the CAPM in the sense that in the end due to assumed perfect diversification only the covariance risk with the market is relevant. The result is the CAPM certainty equivalent valuation (Rubinstein, 1973). This version of the model, which was already suggested by Spremann (2004), shows that our approach is principally so general that it covers the standard project and company valuation theory.

To analyze the NPV of the investment leading to the cash flow X at time t, one simply needs to subtract the initial payment at time 0 from the value according to (3). This is different if we change the perspective to the input-oriented view.

3.2. Input-oriented view

In contrast to the output oriented view, we now additionally consider the initial payment to be made at time 0 (the input). We assume this input to be an amount of money x_0, which can simply be the capital expenditures i_0. If we apply an equity-oriented perspective x_0 could also be the initial equity amount necessary to finance the project. We explore the latter perspective in more detail in the next section.

The economic idea behind the valuation according to the input-oriented view is as follows: Again, we build a reference investment portfolio, to replicate the risk $\rho(X)$ of the cash flow as in the output-oriented view. However, the second condition now is to proceed with the same input x_0 as necessary for the investment leading to cash flow X. To come to a valuation, we additionally assume the following attitude of the decision maker. He or she demands a risk-adequate expected value for the input, derived from market conditions as the reference investment uses the market portfolio, but is risk neutral above the demanded expected value. This implies indifference at time t between the riskfree investment outcome $v(1+r)$ and $X-(y(1+r)+z(1+R_M))$ as long as $v(1+r) = \mathrm{E}(X) - \mathrm{E}(y(1+r)+z(1+R_M))$. In this view, v is the NPV of the project in the sense that it measures today's value of the extra final wealth generated by the project as compared to a capital market investment. This is exactly the main interpretation of the NPV.

The valuation of the cash flow now is given by $V(X) = v + x_0$, i.e. we add the input to the NPV. Summarizing, the three formulae constitute this valuation approach:

$$x_0 = y + z, \tag{5}$$

$$\rho(X) = \rho(y(1+r)+z(1+R_M)), \tag{6}$$

$$\mathrm{E}(X) = \mathrm{E}(v(1+r)+y(1+r)+z(1+R_M)). \tag{7}$$

As above, we exclude short positions in the market portfolio, i.e. $z \geq 0$. The following theorem provides the solution.

Theorem 2 *If the risk measure ρ fulfills PH and TI and $\rho(X) > -x_0(1+r)$ and $\rho(R_M) > -r$,* *then today's value of X according to the input-oriented view is given by:*

$$V(X) = \frac{E(X) - \rho\left(X - x_0(1+r)\right)\frac{\mu_M - r}{\rho(R_M - r)}}{1+r} \tag{8}$$

Proof. Again, the solution can be found by rearranging the replication equations, here (5) to (7). If we solve (5) for y and plug the result into (6), we receive:

$$\rho(X) = \rho\left(x_0(1+r) + z(R_M - r)\right).$$

By applying the properties TI and PH, and utilizing the assumption $z \geq 0$, which is ensured by $\rho\left(X - x_0(1+r)\right) > 0$ and $\rho\left(R_M - r\right) > 0$, we gain:

$$z = \frac{\rho\left(X - x_0(1+r)\right)}{\rho\left(R_M - r\right)}.$$

By plugging z into (5), we can now derive:

$$y = \frac{x_0 \rho\left(R_M - r\right) - \rho\left(X - x_0(1+r)\right)}{\rho\left(R_M - r\right)}.$$

From (7) we have

$$v = \frac{\mathrm{E}(X) - z(1+\mu_M) - y(1+r)}{1+r}.$$

Summing up v, y and z establishes the claim. □

Note that for PI risk measures we have

$$\rho\left(X - \mathrm{E}(X)\right) = \rho\left(X - x_0(1+r)\right) = \rho(X)$$

and the valuation formulae of Theorem 3.1 and 3.2 coincide.

In an equity-oriented view, where X may be the free cash flow to equity (FCFE), the investor can be assumed to design the FCFE in a way that $\rho(X) = 0$, which, interpreted as VaR, means that the worst case the investor would accept is to end up with a cash flow of 0 in the end, implying limited liability up to x_0.

Note that Theorem 3.2 can be seen as an 'in-between' result as it explicitly considers the input needed, which points towards capital market imperfections. However, the interest rate used is still the riskless one, although it would be more suitable to use the risk-adjusted interest rate for defaultable debt claims. Additionally, there are combinations of x_0 and X which cannot be replicated for some risk measures, for instance a riskfree cash flow of $X = 5$ (for any state of the world) with $x_0 = 4$ and $r = 0.05$ has a VaR of -5, which contradicts the prerequisite $\rho(X) > -x_0(1+r)$. Due to these shortcomings we move on to a more realistic treatment of capital market imperfections, which at the same time is the next suggestion of how to value real cash flows.

3.3. Input-oriented view with financing restrictions

Now we assume that the investment leading to the cash flow implies initial capital expenditures amounting to i_0. The decision maker is subject to financing restrictions, meaning that if he or she borrows money the default probability will explicitly influence the interest rate to be paid. We now restrict ourselves to an equity perspective. To this end, we distinguish between the original cash flow Z generated by the project and the cash flow to equity X to be valuated. Note that we exclude taxes for the remainder of this section for simplicity.

From the equity holders' point of view it is advisable to use as much debt as possible. If the decisions maker sets a probability of default (PD) of p, then the lender will demand an interest rate of r_p. The amount of debt granted is at maximum such that the interest rate and the payback can be made with a probability of $1-p$ defining the maximum debt capacity.

$$D_p = \frac{Q_p(Z)}{1+r_p}, \tag{9}$$

where $Q_p(Z)$ is the p quantile of the distribution of Z. Note that we implicitly assume that the debt amount can be even higher than the initial capital expenditure i_0 of the project. By following this view the risk measure chosen is the VaR, i.e.

$$\rho(.) := \mathrm{VaR}_{1-p}(.) = -Q_p(.).$$

For didactic reasons however, we stick to the quantile representation on the level of Z in the following. The initial payment x_0 for the equity holder is the necessary equity amount:

$$x_0 = i_0 - D_p = i_0 - \frac{Q_p(Z)}{1+r_p}\,. \tag{10}$$

Up to here we have assumed that the distribution of Z is located far right from zero so that $Q_p(Z)$ indeed becomes positive and thus makes the cash flow lendable. However, the case $Q_p(Z) < 0$ cannot be neglected. In a one-period world no debt financing will be available for such cases. Thus, one could simply set $x_0 = i_0$. In fact, one has to put even more equity aside if we assume that there are still other stakeholders like suppliers or employees, who may suffer from such a default. To limit the probability for such kind of a default to p, additional equity has to be provided. Thus we have:

$$x_0 = i_0 - \frac{Q_p(Z)}{1+r}\,. \tag{11}$$

Note that the discount factor in (11) has to be modified compared to (10), because the additional equity amount needs to be invested risklessly (at time 0) so that at time t the whole amount $-Q_p(Z)$ is available for buffering the negative outcomes. The investment itself can still be profitable if the expected value of X is highly positive. Finally, it has to be emphasized that this case will be of special importance if we consider multi-period streams of cash flows, which could have a high level of risk with $Q_p(Z) < 0$ at some points in time but are intended to be held longer than up to these cash flows.

The FCFE X at time t can, in both of the above cases, be expressed as $X = Z - Q_p(Z)$, if Z is high enough and no default occurs, and 0 otherwise, because equity has limited liability (in case of $Q_p(Z) < 0$ the liability is limited to exactly minus one times this amount, which on the other hand can be paid back at t if it is not used due to a favorable outcome). Summarizing we have:

$$X = \max(Z - Q_p(Z), 0) \tag{12}$$

with expectation

$$\mathrm{E}(X) = (1-p)\left(\mathrm{E}(Z|Z > Q_p(Z)) - Q_p(Z)\right) \tag{13}$$

and risk

$$\rho(X) = \mathrm{VaR}_{1-p}(X) = \max(\mathrm{VaR}_{1-p}(Z - Q_p(Z)), 0) = 0\,. \tag{14}$$

Note that the interest rate to be paid for borrowing is stochastic in the following sense: The maximum interest rate r_p is paid with probability $1-p$, while with probability p less than this maximum is paid, which we interpret as default.[2] We use the interest rate for a loan with default probability p denoted by the symbol R_D (with expected value μ_D) only in the replication portfolio.

In light of these considerations, the valuation of the risky FCFE X explicitly account for r and R_D. Equations (6) and (7) of Theorem 3.2 are to be replaced by:

$$0 = \rho(X) = \rho\left(y(1+R_D) + z(1+R_M)\right) \tag{15}$$

$$\mathrm{E}(X) = \mathrm{E}\left(v(1+r) + y(1+R_D) + z(1+R_M)\right), \tag{16}$$

where we still assume $z \geq 0$. The following theorem states the valuation result, as long as $x_0 \geq 0$.

Theorem 3 *If the risk measure is $\rho(.) = \mathrm{VaR}_{1-p}(.)$ and R_M has a continuous distribution with $\mu_M > r$ and $\rho(R_M) > -r_p$, then for $x_0 \geq 0$ today's value of the equity cash flows X according to the input-oriented perspective with financing restrictions is given by:*

$$V(X) = \frac{E(X) - x_0 \frac{(1+r_p)(\mu_M - \mu_D) + \rho(R_M - r_p)(\mu_D - r)}{\rho(R_M - r_p)}}{1+r} \tag{17}$$

Proof. The proof is similar to that of Theorem 3.2, i.e. the result can be derived by rearranging the equations (5), (15) and (16). If $x_0 = 0$, then we have $y = -z$ and the assumption of $y < 0$ would lead to $\rho(R_M) = -r_p$, which is a contradiction to the prerequisite $\rho(R_M) > -r_p$. Therefore, we have $y = z = 0$ and the claim is a direct consequence of (16).

[2] We abstain from modeling the exact distribution of R_D below r_p. However, the lowest possible value is -100% corresponding to a total loss of interest and nominal.

If $x_0 > 0$, $0 = \rho\left(y(1+R_D)+z(1+R_M)\right)$ implies $z > 0$. The central difference to the proof of Theorem 3.2 lies in the fact, that due to the stochasticity of R_D the TI property cannot be applied directly to (15). However, the additivity also hold here, as $R_D = r_p$, if $R_M > r_p - \frac{y+z}{z}(1+r_p)$, and $R_D \leq r_p$ (default) otherwise. Thus we have

$$\mathrm{P}\left(R_M \leq r_p - \frac{y+z}{z}(1+r_p)\right) = p\,,$$

as well as $\rho\left(y(1+R_D)\right) = -y(1+r_p)$ and

$$\rho\left(z(1+R_M)\right) = -z(1+r_p) + (y+z)(1+r_p)\,.$$

With these results (15) can be transformed into

$$0 = z\rho\left(1+R_M\right) - y(1+r_p)\,.$$

Now we use the relation $y = x_0 - z$, which is directly implied by (5), and receive:

$$z = \frac{x_0(1+r_p)}{\rho\left(R_M - r_p\right)}\,. \tag{18}$$

From this observation we can derive:

$$y = x_0 - z = x_0 \frac{\rho\left(R_M - r_p\right) - (1+r_p)}{\rho\left(R_M - r_p\right)}\,. \tag{19}$$

Equation (16) now yields:

$$v = \frac{\mathrm{E}(X) - z(1+\mu_M) - y(1+\mu_D)}{1+r}\,.$$

As the value is given by $V(X) = v + x_0 = v + y + z$, we sum up v, y and z and obtain

$$V(X) = \frac{\mathrm{E}(X) - z(\mu_M - r) - y(\mu_D - r)}{1+r}\,.$$

If we now plug in (18) and (19), the claim is proven. □

Note that if the debt interest rate $r_p > r$ is chosen in a way that $\mu_D = r$ (which corresponds to the case of a risk neutral bank or lender), then the term multiplied with x_0 in the numerator simplifies to

$$\lambda^{I,p} = \frac{(1+r_p)(\mu_M - r)}{\rho\,(R_M - r_p)}\,. \tag{20}$$

As the theorem builds on the condition $x_0 \geq 0$, the question arises concerning how to value an investment with $x_0 < 0$. This case correspond to a cash flow producing such a high level of debt capacity that the initial capital expenditure is fully paid by the amount of debt which can be taken on the cash flow. In such a case, the following assumption appears to be the only rational way to consider the situation.

Assumption 1 *If $x_0 < 0$, then the value of X is given by $V(X) = \frac{E(X)}{1+r}$.*

The rationale behind this is that for receiving the FCFE X at time t, we do not need to invest a negative equity payment at time 0. Instead we receive a positive payment of $-x_0$ today. The idea is to immediately cash in this payment and, furthermore, to value the cash flow with an initial equity payment of 0, which is covered by Theorem 3.3 and implies risk neutral valuation. The NPV of such a cash flow is $\frac{\mathrm{E}(X)}{1+r} - x_0$, which is in this case higher than the valuation of X.

4. The multi-period case

Having clarified the question of how to value a cash flow in a one-period model we now move on to the actual problem of interest, namely the case of valuing a stream of risky cash flows (FCFs or FCFEs) $X_1, \ldots, X_T$, which is available upon an initial capital expenditure of i_0 at time 0. The cash flow X_T could indeed simply be the last one in a time-limited project. If the project principally is not time-limited, then the usual business-appraisal practice of establishing two or three phases can be applied with a terminal value at period T. Based on one representative period for the remote future one can use the Gordon-Shapiro model of calculating the present value of a (growing) perpetuity for estimating the expected value. However, concerning the risk of this terminal value X_T additional assumptions have to be made. Another way is to employ multiple techniques in order to estimate the exit price X_T, which becomes then subject to two sources of risk, namely the cash flow or EBIT of the penultimate period and the stochastic multiple factor, which is drawn from the market at time T.

In the following, we restrict ourselves to the output-oriented view stated in Theorem 3.1 and the input-oriented view with financing restrictions according to Theorem 3.3. Theorem 3.2 is regarded as being rather of didactic interest as it is a preparatory step for stating and proving Theorem 3.3.

Before we tackle the problem of valuing the entire stream of cash flows, we first need to clarify how to value a cash flow in a one-period model but with a time interval of variable length. Afterwards we construct two different methods which apply the results for one-period valuation to solve the actual problem of interest.

4.1. Multi-period risk premia

Again, we consider a cash flow located at time t, but now vary this point in time. Without loss of generality we now assume that the time is measured in years, and the interest rates r and r_p are provided as p.a. quantities. Then the valuation results of Theorem 3.1 and Theorem 3.3 can be written in the following form

$$V(X_t) = \frac{\mathrm{E}(X_t) - R(X_t)\lambda_t}{(1+r)^t}\,. \tag{21}$$

Here the symbol $R(X_t)$ stands for $\rho\,(X_t - \mathrm{E}(X_t))$ in case of Theorem 3.1 and x_0 in case of Theorem 3.3. The symbol λ_t corresponds to (4) and (20) in case of Theorem 3.1 and 3.3, respectively. Note that, for Theorem 3.3 we hence use the special case $\mu_D = r$ and for Theorem 3.1 we restrict to the VaR as the risk measure of choice, both of which is not compulsory, but eases the presentation of the complex considerations below.

Furthermore, it has to be stated that in (4) and (20) the symbols r, r_p, R_M and μ_M are to be replaced by their multi-period equivalents. This requires some kind of annualization of these quantities. Additionally, some distributional assumptions need to be made. One straight-forward way for this is modeling the log-return of the market portfolio as a Brownian Motion with drift, implying an easy annualization of the parameters by multiplying t, and then transforming the log-returns into the simple returns used in Theorem 3.1 and 3.3. The advantages of doing so are that the log-returns corresponding to a long time interval can be assumed as normally distributed, leading to a simple return stopping at -100%. This would not be the case if the simple return was assumed to be normally distributed, which implies that in the long run the VaR values are biased. Second, the fact that on a daily, weekly or monthly basis log-returns are proven not to be normally distributed (Aparicio and Estrada, 2001; Bamberg and Neuhierl, 2010) does not apply in the long run as the central limit theorem can be applied to the sum of short-term log-returns. However, there are many other possibilities for modeling the return dynamics of the market portfolio, which we choose not to pursue here, in order to keep the presentation of our ideas as clear as possible.

Let μ'_M and σ'_M symbolize the expected value and the standard deviation of the market portfolio's one-year log-return R'_M and let $r' = \ln(1+r)$, $r'_p = \ln(1+r_p)$. The following theorem provides explicit formulae for λ_t^O and $\lambda_t^{I,p}$.

Theorem 4 *The risk premia according to Theorem 3.1 and 3.3 corresponding to time* $t > 1$ *under the assumption of a normally distributed market-portfolio log-return* R'_M *and for* $\rho(.) = \mathrm{VaR}_{1-p}(.)$ *are*

$$\lambda_t^O = \frac{\exp\left(t\mu'_M + t{\sigma'_M}^2/2\right) - \exp(tr')}{\exp\left(t\mu'_M + t{\sigma'_M}^2/2\right) - \exp(t\mu'_M + q_p\sqrt{t}\sigma'_M)}, \tag{22}$$

where q_p *is the q-quantile of the standard normal distribution, and*

$$\lambda_t^{I,p} = \frac{\exp\left(t\mu'_M + t{\sigma'_M}^2/2\right) - \exp(tr')}{1 - \exp\left(t\mu'_M + q_p\sqrt{t}\sigma'_M - tr'_p\right)}. \tag{23}$$

Proof. Both statements can be shown by plugging the expected value of the simple return given the parameters of the normally distributed log-return (see Dorfleitner, 2003, formula (3.1)) and the annualization of μ'_M, ${\sigma'}^2_M$, r and r_p by multiplication with t into formulae (4) and (20). □

Note that the numerators of both risk premia are always positive as long as $\mu'_M > r'$, which is a standard assumption, without which no investments in stocks would take place. However, for

$t \to \infty$ we have $\lambda_t^O \to 1$. This can be interpreted in the following way. For cash flows lying far in the future the numerator of (3) goes to $-\rho\,(X_t)$, i.e. such cash flows only provide a positive value contribution if $\rho\,(X_t) < 0$ implying that the distribution of X_t lies so far on the right hand side of zero that the p-quantile is positive.

For $\mu'_M > r'_p > r'$, the risk premia $\lambda_t^{I,p}$ become higher than any finite number as t is increased. However for $t \to \infty$ the nominator finally becomes negative which implies a pole somewhere in between. This means that the condition $\rho\,(R_M) > -r_p$ of Theorem 3.3 is violated and naturally provides a degree of limitation of the input-oriented approach with financing restrictions as it cannot be used in these cases, at least not in combination with the Brownian Motion model for the market portfolio log-return. Therefore, practically, the instant of time T of the last cash flow has to be lower than that value of t at which the sign change occurs.

4.2. Two or more cash flows and the cumulation principle

The first suggestion of valuing a stream of cash flows is simply to cumulate all the cash flows up to the final instant of time T. The resulting super cash flow is then valued by a one-period model with $t = T$ and the corresponding multi-period risk premium λ_T. The idea behind cumulation is that risky cash flows, realizing in period $t(< T)$, are passed on to time T with the riskless interest rate. In the case of a positive cash flow the money is invested at interest rate r and in the case of a negative cash flow a loan is taken at r. Such modeling implies an infinite risk bearing capacity at any point of time t. At the end of the valuation interval $[0,T]$, the accumulated stochastic time value of all cash flows is valued with a one-period model.

Formally we have:

$$V(X_1,\ldots,X_T) = \frac{\mathrm{E}\left(\sum_t X_t(1+r)^{T-t}\right) - R\left(\sum_t X_t(1+r)^{T-t}\right)\lambda_T}{(1+r)^T}\,. \tag{24}$$

Note that this due to the PH property of R and the expectation operator be transformed into a representation which displays the formulae as functions of the stochastic PV. According to Hazen

(2009), the calculation of a stochastic PV is quite common for some decision makers, a view which thereby is also covered by our approach.

4.3. Two or more cash flows and the separate valuation principle

While the cumulation approach represents an idealized extreme case, the opposite also is possible and even more realistic, namely to take the risk of every cash flow X_t exactly at time t, i.e. to value each cash flow separately and to add up the T values. This implies that no balancing whatsoever can be implemented from any time t to a later period.

The separate valuation implies inter-temporal value additivity, which obviously does not hold for the cumulation method, but for most standard valuation principles (like CAPM-based DCF methods). The resulting valuation of the stream of cash flows is

$$V(X_1,\ldots,X_T)=\sum_{t=1}^{T}\frac{\mathrm{E}(X_t)-R(X_t)\,\lambda_p^t}{(1+r)^t}\,. \tag{25}$$

As can be seen from (25), in each period a certainty equivalent is calculated and then discounted at the riskless rate. This approach, which is also called the certainty equivalent method (CEM), a notion coined by Kudla (1980), who references Robichek and Myers (1966a) but does not determine the certainty equivalent objectively. In the case of the output-oriented perspective, this is easily implementable for any initial capital expenditure i_0 as no direct relation between i_0 and the valuation of each cash flow according to Theorem 3.1 exists.

However, in the case of input-orientation for each X_t a value $x_0^{(t)}$ has to be found. To achieve these values, the amount i_0 needs to be allocated to the periods. Let $i_0^{(t)}$ symbolize the amount corresponding to t. Then the allocation is subject to

$$i_0=\sum_{t=1}^{T}i_0^{(t)}\,. \tag{26}$$

The values $i_0^{(t)}$ can be restricted to being ≥ 0, but this is not a necessary condition. Independently of the i_0 allocation, the overall equity capital needed is determined by

$$x_0 = i_0 - \sum_{t=1}^{T} \frac{Q_p(Z_t)}{(1+\bar{r}_t)^t}\,, \tag{27}$$

where $\bar{r}_t = r$ in the case of a negative Q_p and $\bar{r}_t = r_p$ in the case of a positive Q_p. Once an $i_0^{(t)}$ is found for a period t then $x_0^{(t)}$ (and vice versa) can be calculated as

$$x_0^{(t)} = i_0^{(t)} - \frac{Q_p(Z_t)}{(1+\bar{r}_t)^t}\,. \tag{28}$$

Different allocations can be considered. However, as the valuation result can heavily depend on the concrete allocation, some additional assumptions and economic rationale need to be applied. As long as $\exp(-tr')\lambda_t^{I,p}$ increases monotonously in t, which should be the case for sensible parameter values, we receive the value maximizing (minimizing) allocation, in case of $i_0^{(t)} \geq 0$, by setting $i_0^{(1)} = i_0$ ($i_0^{(T)} = i_0$) and all other values equal to zero. Yet, these allocations are difficult to justify and appear to be completely arbitrary. Theoretically, the allocation could be generated from a depreciation scheme for the initial capital expenditure, but again a fast depreciation would increase value. The only economically sound way of solving the problem in our view is to account for the duration of the capital lockup, i.e. capital is locked up in the project until it gets repaid through the cash flows of the project. We implement this in two ways, based on the lockup of i_0 and of x_0.

Capital lockup of i_0. We consider the expected values of the original cash flows $Z_1,\ldots,Z_T$ as these are acquired through investing i_0. The rationale is that i_0 is to be paid back continuously by the cash flows of the project. However, since the cash flows themselves are stochastic, we substitute them with their expected value. The following algorithm results:

1. Start with $t = 1$.
2. If $\sum_{s=1}^{t-1} i_s < i_0$, set $i_t = \min\{\mathrm{E}(Z_t), i_0 - \sum_{s=1}^{t-1} i_s\}$, otherwise $i_t = 0$.
3. Increment t. If $t \leq T$ then go to 2, otherwise stop.

If $\sum_{s=1}^{T} i_s < i_0$, then we set $i_T = i_0 - \sum_{s=1}^{T-1} i_s$. In this case, however, it is impossible to achieve a positive NPV. The $x_0^{(t)}$ are calculated according to (28). Note that it is generally possible to have negative $x_0^{(t)}$ for some t.

Capital lockup of x_0. The second procedure follows a similar idea, however based on the $X_1, \dots, X_T$ and allocates x_0 directly to $x_0^{(1)}, \dots, x_0^{(T)}$ with the restriction $x_0^{(t)} \geq 0$. The values for $i_0^{(1)}, \dots, i_0^{(T)}$ can be calculated subsequently. The determination of $x_0^{(t)}$ follows the principle of 'equity payback', but with accounting for later capital needs for buffering negative cash flows. The x_0 is allocated according to the expected lockup period, i.e. according to the positive expected X_t. Whenever $\mathrm{E}(X_t) \leq 0$, no equity can be paid back. The last period T is allocated the remainder of the entire equity capital not repaid up to $T-1$ regardless of $\mathrm{E}(X_T)$. The following algorithm results:

1. As a preparation step to account for the capital needs in every period set $x_0^{(t)} := \max\left\{0, \frac{-Q_p(Z_t)}{(1+r)^t}\right\}$ for all $t = 1, \dots, T$.
2. Set $t = 1$.
3. If $\mathrm{E}(X_t) \leq 0$, then $x_0^{(t)} := 0$.
4. If $\mathrm{E}(X_t) > 0$ then $x_0^{(t)} := \min\left\{\mathrm{E}(X_t), x_0 - \sum_{s=1}^{T} x_0^{(s)}\right\}$.
5. Increment t. If $t < T$, then go to 3.
6. Set $x_0^{(T)} := x_0 - \sum_{s=1}^{T-1} x_0^{(s)}$.

Note that in step 4 the FCFE payback is limited if in periods $s > t$, equity capital is needed to buffer a positive VaR. Besides this restriction the equity is locked up for as short a time as possible as it is costly, and thereby value-diminishing if employed for a longer period than necessary.

Note that for the case that $\lambda_t^{I,p}$ exceeds every boundary, an FCFE X_t with $\mathrm{E}(X_t) > 0$ lying far in the future can only create a positive value contribution if $x_0^{(t)} \leq 0$. Even if the payback of i_0 or x_0 is ensured by high cash flows in earlier periods, the positive value contribution will only be achieved if $Q_p(Z_t) \geq 0$.

4.4. Implementation of the cumulation approach as optimal partial cumulation

From an economic viewpoint the advantage of the cumulation approach described above, namely the unlimited risk capacity in between, should lead to a higher and not a lower value compared to the CEM. In fact, this effect does not always present itself as the lower risk of the cumulated cash flow (as compared to the sum of the separate cash flows' risks) can be over-compensated by the higher risk premium charged at time T. For this reason it is economically plausible to assume that the decision maker will implement cumulation only in order to end up with a higher value than that achieved by the CEM.

Thus, according to the concept of optimal partial cumulation we suggest to only use the unlimited risk capacity in between to increase value. This means that a cash flow may be passed on to $t+1$ for some t, but not for all of the instants of time $1,\dots,T-1$, at which this is possible. Overall, 2^{T-1} possibilities of passing the cash flow through exist, corresponding to the cardinal number of the power set of the $T-1$ instants of time. As an example, let $T=3$. One could now choose among four possibilities, namely 1;2;3, 1-2-3, 1;2-3 and 1-2;3, where the symbol '-' depicts the passing on to the next period. The variant 1;2-3, for instance, implies that the valuation is done as the sum of the valuation of the cash flow at time $t=1$ and the valuation of the cumulated cash flow from the periods 2 and 3 which realizes at $t=3$. The procedure is obviously a generalization of the CEM and the cumulation method as it comprises both possibilities. The optimal partial cumulation is achieved by calculating all 2^{T-1} possibilities and then choosing the one which yields the highest value. Thus, the resulting valuation is at least as high as the CEM value. Note that if we combine the optimal partial cumulation with the input-oriented perspective according to Theorem 3.3 then each of the 2^{T-1} variants requires an allocation of x_0, however with less than T relevant periods.

4.5. Assumptions underlying the different variants and discussion of decision theoretic aspects

This subsection is dedicated to a discussion of the question which assumptions implicitly hold in each of the different variants of our approach. Among those items which can be chosen in one way or another we can distinguish those due to individual preferences and those due to modeling the real world and its possibilities.

Individual preference choices. The risk measure and the perspective (output-oriented or input-oriented in the two variants) are the two parts of the approach which can be adapted to represent individual views and preferences. Additionally, to an extent also the choice of the reference investment is subjective. The risk measure should (respectively the probability p in case of the input-oriented perspective) be chosen to fit the decision maker's perception of risk best. Our approach does not require or propose certain risk measures as for instance the coherent ones. The output-oriented perspective is apt for those investors that mainly look at the future cash flows and the risk of these, while the input-oriented view pronounces today's capital needs more and is at the same time partly risk neutral concerning the future cash flows above the reference investment's expected value after establishing the same risk. The rate of substitution between the risk and the expected value is *not* individually determinable as this figure is implied by the distributional properties of the reference investment. The rationale behind this is an opportunity costs argument stating that the decision maker adjusts his or her risk-return expectations to the possible real world investment alternative. Summarizing, there are some individual choices that can be made, but surely less as compared to most utility function based approaches. The two different allocations schemes for x_0 in the input-oriented view with financing restrictions apply two slightly different assumptions concerning the payout policy. Both are risk-neutral concerning the payout, but the i_0 lockup focuses on the paypack of i_0 and subtracts debt capital as discounted values, while the second method is concerned about the paypack of equity directly.

Real world modeling. As pointed out above, the reference investment in the capital markets imposes the valuation to a large part. Its choice is subjective, but once it is chosen, the used stock market model has consequences for the calculation of lambda. Formulae (22) and (23) are only a suggestion and will look differently if the stock market model is chosen in a more sophisticated way. Moreover, the possibility to cumulate is also attributable to modeling the decision maker's conditions. If the cash flows are small enough and the credit lines or the liquidity are high enough that cumulation can be regarded as realistic, then this variant of the model can be applied. If this is done in a value-oriented way and therefore by the optimal partial cumulation then the certainty equivalent method also is enfolded. Generally, behind all variants there is the assumption of value additivity with respect to time. The economic rationale is that the valuing individual appraises a bundle of cash flows equally to the sum of each cash flow's value. This property is beyond doubt in standard valuation approaches like arbitrage free pricing or discounted cash flow. Only very few known approaches do not fulfill this property. In our context, once the decision maker has accepted to imperfectly replicate each single cash flow of the stream with an own reference investment, the time-additivity directly comes from the portfolio additivity applicable to the reference investments consisting of real world securities. Note that the value-additivity-over-time assumption is even present if optimal partial cumulation leads to the extreme case of full cumulation, as it is used in the optimization process.

Summarizing, there are some choices with which individual preferences can be represented. The resulting time and risk preferences are, however, implied not only by these choices but much more by the properties of the reference investment (expected returns and risk of the stock market and the riskless interest rate resp. the interest rate for default probability p). In any case for a sensible setup of the corresponding parameters, all of the variants express risk aversion in the sense that given two cash flows A and B with identical expected value (and initial capital need in the input-oriented variants) and A having a lower risk than B, then A gains a higher valuation than B, while for two identical cash flows at two different instants of time the earlier one always will be valued higher.

By assuming that the valuation is based on μ-risk reasoning we account for the fact that capital markets are neither complete nor perfect. Therefore, it would not be appropriate to discuss our approach within a framework that assumes perfect and complete capital markets. In particular, the assumption of no arbitrage is not valid within and also not necessary in our framework. Even if capital markets were free of arbitrage, which is questionable in practice (see e.g. Shleifer (2000)), the no arbitrage concept would only be applicable if the risky cash flows were tradeable (long and short) on a market. This is neither realistic for a specific project of a company nor for non-listed

SMEs.

5. An example based on Monte Carlo simulation

In the following, we illustrate the theory developed above with a virtual but realistic example. A great advantage of risk-value model valuation is that MC simulation techniques can be utilized quite naturally. Although, principally it would be possible to use multi-variate distributions to model the stochasticity of the cash flows and to derive closed form solutions, the MC simulation appears to be the most flexible way in which to depict informations the decision maker (or the company) has. One can directly describe cash flows dependent on risk factors or one can model future balance sheets and future accounting data of the company and derive the cash flows from these. Indeed, with the ever increasing computing power, MC simulations are being increasingly performed in the investment calculation process of companies (Graham and Harvey, 2001).

We consider a four-period project with an initial capital expenditure of 25 million, whose cash flows are influenced by three identifiable risk factors. Let the default probability, which is to be met in every of the four periods, be $p = 1\%$ (which more or less corresponds to a BB rating). We assume a flat interest rate term structure with $r = 2.1\%$. The risky-debt interest rate of a risk-neutral bank is then $r_p = 3.13\%$. The FCF Z_t in period $t \in \{1,2,3\}$ can, in dependence of the risk factors $RF1_t$ (a commodity price), $RF2_t$ (an exchange rate), $RF3_t$ (a stock index) be expressed as:

$$Z_t = (100{,}000 + RF3_t \cdot 30 + \varepsilon_t^{ns}) \cdot (ps \cdot RF2_t - 0.2 \cdot RF1_t) - 800{,}000 + \varepsilon_t, \tag{29}$$

with ps (here: 66) symbolizing a selling price of the product, $\varepsilon_t^{ns} \sim \mathcal{N}(0;10{,}000)$ symbolizing the error quantity of the number of units sold (the first of the terms in brackets, which is also influenced by $RF3_t$) and $\varepsilon_t \sim \mathcal{N}(0;10{,}000)$ denoting a general error term. In period 4 the project is sold at a stochastic price which will be within a range of 0.5 to 11 million, in which we assume a uniform distribution. Costs of 1 million will accrue in any case. This leads to an FCF amounting to a quantity, which is uniformly distributed in the interval $[-0.5, 10]$ million and assumed to be independent of the risk factors and error terms. While both error variables are assumed to be independent of each other and also independent of the risk factors, the latter are correlated. Table 1 displays the parameter values of the risk factors' log-returns (i.e. $\ln(RF_{t+1}/RF_t)$), which are assumed to be multi-variately normally distributed and independent from one period to another.

With the assumption that $RF3$ represents the market portfolio, we can calculate the risk premia according to (22) and (23). The values are displayed in Table 2.

Now we perform a simulation of $Z_1,\dots,Z_4$ with 10,000 runs. Table 3 displays the expected values of the FCFs Z_t and the FCFE X_t and the $Q_p(Z_t)$ values estimated from the simulation results as well as the x_0 allocations, required for the two suggested methods in the input-oriented CEM. For the purposes of this example we regard the number of runs as high enough, which can be verified by the simulated value for $Q_p(Z_4)$, while the analytical value equals $-395{,}000$.[3]

The valuation results are presented in Table 4. Notably, in both perspectives the cumulation approach produces a negative NPV despite its low risk values, which is attributable to the fact that the lambda values (cf. Table 2) increase steeply with time. This effect, which is less pronounced in the output-oriented perspective, explains the lower difference to the CEM. All of the CEM variants produce a positive NPV. Among these, the output-oriented one yields the lowest valuation, which could be explained by the fact that as the NPV is positive a certain amount of the expected value is valued risk-neutral (according to (16)) in the input-oriented view, which overcompensates the higher λ-values. Notably, neither method of x_0 allocation produces results that differ much, which is not surprising as the allocations themselves (see Table 3) do not differ much either.

If, additionally, the optimal partial cumulation is used, it turns out that the zero correlation between the cash flows at $t = 3$ and $t = 4$ is so advantageous that indeed the optimal partial cumulation is 1;2;3-4. This holds true for the input-oriented and the output-oriented view, resulting in the NPVs of 907,005 (output-oriented view), 1,519,668 (input-oriented view with the capital lockup of i_0 method) and 1,505,306 (input-oriented view with x_0 payback method). The additional possibilities

[3] If only this single cash flow were considered, no simulation would be necessary. However, the simulation of Z_4 aims at covering the interdependence with the other cash flows.

obviously increase the value in this example.

To compare our approach with the standard DCF approach we discount the expected values of the FCF with unlevered capital costs of 7.62%, which corresponds to an unlevered beta estimate amounting to 1.32.[4] The resulting value of the project is 26,958,170 with an NPV of 1,958,170, thus representing a much higher figure. Here, the correlation with the market is relatively high. Yet, a lower correlation with the market portfolio would lead to an even higher value.

6. Practical aspects and further considerations

Next, we discuss various relevant issues for implementing the suggested model in its several variants in practice and also include considerations concerning further research.

6.1. Hedging, diversification and the riskyness of the cash flows

Above, the stream of cash flows $X_1,\dots,X_T$ to be valued is introduced as given. However, in practice it cannot in any case be left as it is. Simple risk-reducing hedging possibilities which are already available today need to be implemented, because in all of the variants above risk reduces value. Consider, for instance, a eurozone company facing a positive cash flow amounting to a fixed sum in USD at $t=1$. As the foreign exchange rate is stochastic ex ante the cash flow X_1 in EUR would be stochastic and any of the risk-value models above would imply a negative risk adjustment of the expected value. If we additionally set up a USD/EUR forward amounting to the initial USD sum (which comes at a price of 0) we can even transform the EUR cash flow at time $t=1$ into a certain one from today's perspective. In this sense, the model makes the value contribution of hedging instruments explicit and allows the calculation of the optimal hedge. Therefore, leaving the cash flows unhedged can result in a value which is too low.[5]

[4] We derive the beta from equalizing the CAPM CEM valuation of the first period's cash flow with its expected value discounted with the risk-adjusted cost of capital. The leverage policy does not play a role as we exclude taxes in this example.

[5] Although the approach clearly is not to be applied in cases in which the stream of cash flows is fully replicable, one can reconsiliate our approach with no arbitrage pricing in the following way (in the one period case). First, consider the

A similar idea concerning the aspect of diversification is applicable. While in the CAPM a fully diversified investor is assumed, the approach presented here so far focuses on a completely undiversified investor. Even if it is realistic for unlisted companies to assume that the valuing individual is not perfectly diversified, other assets held by the decision maker could, in any case, be made allowance for. The same applies for valuing a project within a company, which usually has other cash flow generating projects as well. Gleißner and Wolfrum (2009) already discuss this issue in their setting, which is a special case of our general approach. Technically spoken, in this case only the incremental risk induced by the cash flow should be the quantity measured by $\rho(X)$. In the case of VaR, this measure could be the incremental VaR (IVaR) as, for instance, discussed by Wang (2002).

To derive the probability of default p needed in the input-oriented perspective with financing restrictions, one can simply take the minimum of the maximal values the decision maker (the buyer, the seller, the owner etc.) resp. the creditor are willing to accept. Another issue connected with p is that, given a certain rating, the default probability due to the possibility of rating migrations usually increases over time (see for instance Tsaig et al. (2011)). This is not incorporated in the multi-period framework suggested above as it would make things more complicated. Principally, the approach is general enough to implement a $p(t)$ as function of time as well. However, as the p is more than just a given default probability, the decision maker can still choose to apply a constant p over all time horizons.[6]

approach only for cash flows with non-zero risk. Second, add the shorted replication portfolio (be it static or dynamic) to the original cash flow, thereby hedging it perfectly. Now the future cash flows is zero (and so its risk). Its value is also zero. Third, the price of the replication portfolio is the sum you receive to end up with nothing, thus interpretable as the value of the original cash flow.

[6]It has to be noted that even if p is applied for every of the T periods in the CEM, it does not represent the probability that within the time interval T a default only appears with probability p, as overall T (stochastically dependent) draws are made. Thus, p is rather a parameter describing some sort of preferences. A practical solution can also be to fix p to a value averaged over the PD term structure.

6.2. Leverage policy, taxes and other future decisions of the company

In corporate finance the leverage policy, which addresses the question of how to choose the capital structure over time, is a crucial issue with some consequences for project and business valuation (see Berk and DeMarzo, 2014, ch. 18, for a textbook treatment). Hence, the question arises as to which leverage policies can be covered by our approach.

Apparently, if the cumulation method is applied then there is only one point in time at which debt is to be paid back. Thus one cannot speak of a leverage policy at all. In case of Theorem 3.1 any amount of debt could be realized, whereas in case of Theorem 3.3 the debt volume is determined by the method. It is not possible to choose a higher level as this contradicts the idea of maximizing the debt, but if it is chosen to be lower, and the value decreases due to this (because x_0 increases) then it can be accounted for.

If we consider the CEM in its pure form or in the partial cumulation variants two cases are to be distinguished. In the output-oriented view every leverage policy, be it a constant debt-equity ratio, predetermined debt levels or constant interest coverage, can be implemented and has to be accounted for in the cash flow simulation. In the input-oriented view the debt levels are predetermined by the method. Again, it would be possible to deviate from these downwards.

For simplicity and to present our new approach in maximal clarity we refrained from explicitly modeling taxes although we are conscious of the fact that these are a major issue in business valuation. Since in practice taxes are non-negligible, we recommend to explicitly account for them in the cash flow model. This means that the Z_t, the X_t and the interest amounts paid or received are to be calculated after taxation. Consequently, in the reference investment the interest rates r and r_p and the return of the market portfolio R_M need to be the values after taxes. The debt tax shield is then implicitly considered in our approach. The leverage policy is also accounted for, because the FCFE X_t directly depend on it as taxes clearly affect cash flows. Even the search for an optimal leverage policy can be implemented if the amount of debt is modeled as a variable input to the simulation. However, it is not possible to derive closed form solutions as known from standard corporate finance.

Concerning the payout policy, we used a mild form of introducing decisions in Section 4.4, as we assumed a static procedure, in which all the decisions were fixed ex ante. In general, decisions relating to future instants of time need not be made today. However, the usual way in which to treat such decisions is to set up or at least to search for a stochastic control strategy which describes how the future decisions are made in dependence of the respective state of the world. The control strategy could, for instance, concern the leverage and payout policy or the use of hedging instruments. The determination of such a strategy would be heavily dependent on both the restrictions on the range of possible decisions at each instant of time as well as the modeling of the cash flow dynamics. Once it is found, it could be integrated into the cash flow simulation and thus lead to optimized (but still risky) cash flows. However, in this case, the theoretical framework, especially of the input-oriented approach with financing restrictions, also needs to be adapted with respect to $x_0^{(t)}$. We regard this as a fruitful line of further research. When treating these questions, it will be unavoidable to use the concept of filtrations as is already done by Kruschwitz and Löffler (2005) in the context of classical DCF methods and dynamic leverage policies.

7. Conclusion

On a clear axiomatic basis, we introduce an approach to valuing streams of risky cash flows using risk-value models. It is based on imperfect replication, which has occasionally been treated in literature, but not in the general form pursued here. Contrary to previous literature, we constitute a full multi-period approach for a risk-adjusted valuation of projects and businesses. The appraisal is performed based on the expected value of the output's or input's magnitude and the risk of the (output) cash flow, as captured by a risk measure, which also depicts the risk preference of a (individual or representative) decision maker. In three variants, we derive a valuation formula from equations regarding the μ-ρ equivalence between the cash flow and a reference investment. The approach is so general that it also comprises the CAPM certainty equivalent. Even if the risk premia are higher in the input-oriented approach, it may yield higher valuations of investments with a positive NPV since in this approach a part of the cash flow's expected value is valued risk-neutrally. To derive a multi-period valuation we use the principles of separate valuation plus value-additivity over time and of cumulating the cash flows. In the case of the input-oriented view, the initial capital expenditure has to be allocated to the periods, a procedure for which we suggest two algorithms.

It clearly needs to be emphasized that each of the various variants leads to a theoretical value, which is in line the model assumptions, and not to a verifiable market price. In the light of imperfect real capital markets, there is a difference between price and value and in the consequence also between valuation concepts and asset pricing models. Whether or not a valuation concept is apt for a decision maker depends on the acceptability for him or her of the model's assumptions. In contrast, asset pricing models have to prove their aptitude through their capability of explaining real market prices. From this perspective, we wish to suggest a new class of valuation models to real world decision makers, who are appealed for assessing the question which variant fits their views and needs best.

This article is intended to lay the foundations for a new valuation approach. However, we also suggest several directions for its further development. As the axiomatic basis of our approach is not particularly restrictive and the information requirements relating to the subject and object of appraisal are transparent, the model framework seems to be very promising with regard to inspiring further research and to being applied in real-world valuation setups, especially in SMEs.

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	E	SD	Corr($RF1$,.)	Corr($RF2$,.)	Corr($RF3$,.)
$RF1$	10%	15%	1	−0.05	0.10
$RF2$	0%	8%		1	0.30
$RF3$	4%	20%			1

Table 1
Expected values (E) of, standard deviations (SD) of and correlations between the log-returns of the three risk factors.

t	1	2	3	4
λ_t^O	0.1000	0.1502	0.1916	0.2283
$\lambda_t^{I,p}$	0.1115	0.1800	0.2457	0.3123

Table 2
Lambda values according to formula (22) (λ_t^O) and formula (23) ($\lambda_t^{I,p}$) for the four periods.

t	1	2	3	4	1-2-3-4
$\mathrm{E}(Z_t)$	9,027,217	9,052,032	8,976,509	4,763,549	32,972,733
$\mathrm{E}(X_t)$	7,090,401	8,193,304	9,590,351	5,159,367	26,366,389
$Q_p(Z_t)$	1,943,483	866,686	−602,375	−395,224	6,634,370
Allocation of x_0 according to capital lockup of i_0					Σ
$i_0^{(t)}$	9,027,217	9,052,032	6,920,751	0	25,000,000
$x_0^{(t)}$	7,142,743	8,237,177	7,486,716	363,697	23,230,333
Allocation of x_0 according to equity payback					Σ
$x_0^{(t)}$	7,090,401	8,193,304	7,582,930	363,697	23,230,333

Table 3
Simulated expected values of FCFs (Z_t) and FCFEs (X_t) as well as resulting x_0 allocations.

Output-oriented view		
CEM	$V(X_1,\ldots,X_4)$	25,661,451
	NPV	661,451
Cumulation	$V(X_1,\ldots,X_4)$	24,809,488
	NPV	−190,512
Input-oriented view with financing restrictions		
CEM (i_0 lockup)	$V(X_1,\ldots,X_4)$	24,527,584
	NPV	1,297,251
CEM (x_0 payback)	$V(X_1,\ldots,X_4)$	24,518,664
	NPV	1,288,332
Cumulation	$V(X_1,\ldots,X_4)$	18,763,342
	NPV	−372,070

Table 4
Valuation results for the different variants. The combination of cumulation and the input-oriented view with financing restrictions (last two rows) is calculated with an equity capital need of $x_0 = 19{,}135{,}412$.